

ON THE CONJUGACY SEPARABILITY OF GENERALIZED FREE PRODUCTS OF GROUPS

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It is proved that generalized free product of two finite p -groups is a conjugacy p -separable group if and only if it is residually finite p -groups. This result is then applied to establish some sufficient conditions for conjugacy p -separability of generalized free product of infinite groups.

1. A group G is called conjugacy separable (conjugacy p -separable) group if whenever elements a and b of G are not conjugate in G , there is a homomorphism φ of G onto finite (respectively, finite p -group) X such that elements $a\varphi$ and $b\varphi$ are not conjugate in X .

It is easy to see that a conjugacy separable group is residually finite group and a conjugacy p -separable group is residually finite p -groups. In general inverse statements are not true. But in some cases residually finite group (residually finite p -groups) is a conjugacy separable (respectively, conjugacy p -separable) group too. For example, J. Dyer [7] have proved that a free product with amalgamated subgroups of two finite groups is conjugacy separable group (in 1963 G. Baumslag [5] showed that any such group is residually finite).

Not every free product with amalgamated subgroups of two finite p -groups must be residually finite p -groups. G. Higman [9] have obtained necessary and sufficient conditions for such groups to be residually finite p -groups. The question arises whether these conditions are enough for such group to be a conjugacy p -separable group.

An element g of a group G is called C_{fp} -separable if for any $a \in G$ such that elements a and g are not conjugate in G , there exists a homomorphism φ of G onto finite p -group X such that $a\varphi$ and $g\varphi$ are not conjugate in X . So a group G is conjugacy p -separable if and only if every element $g \in G$ is C_{fp} -separable.

It was proved in [1] that
if a free product with amalgamated subgroups of two finite p -groups is a residually finite p -group then every infinite order element $g \in G$ is C_{fp} -separable.

In fact the following generalization of this statement holds:

Theorem 1. *Suppose $G = (H * K; A = B, \varphi)$ is a free product of two finite p -groups H and K with amalgamated via isomorphism φ subgroups A and B . G is a conjugacy p -separable group if and only if G is a residually finite p -groups.*

Applying this result and using a standard technique we proved the following theorem:

Theorem 2. *Suppose that H and K are conjugacy p -separable groups, $A \leq H$ and $B \leq K$ are central subgroups and every finite p -index subgroup of A and of B is p -separable in H and in K respectively. Then $G = (H * K; A = B, \varphi)$ is a conjugacy p -separable group.*

(Recall that a subset M of a group G is called p -separable if for every $a \in G$, $a \notin M$, there is a homomorphism φ of G onto finite p -group X such that $a\varphi \notin M\varphi$.)

The description of conjugacy p -separable finitely generated nilpotent groups has been given in [2]. It follows from the theorem 2 that the following statement holds:

Theorem 3. *Suppose $G = (H * K; A = B, \varphi)$ is a free product with amalgamated subgroups of two finitely generated nilpotent groups H and K , H and K are conjugacy p -separable groups, A and B are p' -isolated central subgroups of H and K respectively. Then G is a conjugacy p -separable group.*

(Recall that if p is a prime then a subgroup X of a group Y is called p -isolated if $y^p \in X$ implies $y \in X$ for every $y \in Y$. A subgroup X of a group Y is called p' -isolated if X is a q -isolated for every prime $q \neq p$.)

2. The proof of theorem 1 is a certain modification of the J. Dyer's proof of her result in [8].

Graph Γ is a system of two sets $V = V(\Gamma)$ (the set of vertexes) and $E = E(\Gamma)$ (the set of edges) and of three mappings $\sigma : E \rightarrow E$, $\sigma : E \rightarrow V$ and $t : E \rightarrow V$ such that $\sigma(\bar{e}) = t(e)$, $t(\bar{e}) = \sigma(e)$, $\bar{e} \neq e$ and $\bar{\bar{e}} = e$ for every $e \in E$. The edge \bar{e} is called inverse to edge e , the vertex $\sigma(e) \in V$ is called the origin of edge e , the vertex $t(e) \in V$ is called the end of edge $e \in E$. If $\sigma(e) = u$ and $t(e) = v$ then we write $e = (u, v)$.

The group graph is a pair (\mathcal{G}, Γ) of connected graph Γ and mapping \mathcal{G} . The mapping \mathcal{G} associates every vertex $v \in V(\Gamma)$ with group G_v

and every edge $e \in E(\Gamma)$, $e = (u, v)$ with group G_e and two mappings $\rho_e : G_e \rightarrow G_u$ and $\tau_e : G_e \rightarrow G_v$ such that $G_{\bar{e}} = G_e$, $\rho_{\bar{e}} = \tau_e$ and $\tau_{\bar{e}} = \rho_e$. The groups G_v and G_e are called vertex group and edge group of a group graph (\mathcal{G}, Γ) respectively.

Suppose (\mathcal{G}, Γ) is a group graph and T is a maximal tree of Γ . Let X_v be a set of generators and R_v be a set of relations of the vertex group G_v for every vertex $v \in V = V(\Gamma)$ (and if $v_1 \neq v_2$ then $X_{v_1} \cap X_{v_2} = \emptyset$). The fundamental group $\pi(\mathcal{G}, \Gamma)$ of a group graph (\mathcal{G}, Γ) is a group with generators $\bigcup_{v \in V} X_v$ and t_e , $e \in E(\Gamma) \setminus E(T)$, and relations $\bigcup_{v \in V} R_v$ and

$$\begin{aligned} g &= g(\rho_e^{-1}\tau_e), & e \in E(T), \quad g \in G_e\rho_e, \\ t_e^{-1}gt_e &= g(\rho_e^{-1}\tau_e), & e \in E(\Gamma) \setminus E(T), \quad g \in G_e\rho_e, \\ t_{\bar{e}} &= t_e^{-1}, & e \in E(\Gamma) \setminus E(T). \end{aligned}$$

It is possible to prove that the group $\pi(\mathcal{G}, \Gamma)$ does not depend on a choice of vertex groups presentations and maximal tree T .

It is well known ([6, 10, 11]) that every finite extension of a free group is isomorphic to a fundamental group of a group graph with finite vertex groups.

To prove the theorem 1 the following result is also required ([1]):

Proposition 2.1. *Suppose $H \leq G$ is a subnormal finite p -index subgroup. If $h \in H$ is a C_{fp} -separable in group H then h is C_{fp} -separable in group G .*

Now we are ready to prove the theorem 1.

Suppose $G = (H * K; A = B, \varphi)$ is a free product of two finite p -groups H and K with amalgamated via isomorphism φ subgroups A and B . The necessary conditions in theorem are evident.

Let G be a residually finite p -groups. Then ([4, lemma 2.1]) G is an extension of a free group F by finite p -group. In view of mentioned above result from [1] to prove that G is a conjugacy p -separable it is enough to show that whenever a and b are finite order elements and not conjugate in G , there is a homomorphism ψ of G onto a finite p -group X such that $a\varphi$ and $b\varphi$ are not conjugate in X .

Since subgroup of G which is generated by F and a is a subnormal in G it follows from proposition 2.1 that we can assume that group G is generated by F and a . Then the quotient group G/F is cyclic and

therefore if $aF \neq bF$ then natural homomorphism of G onto G/F is required.

Suppose $aF = bF$. Since F is a torsion-free so the orders of a and b are equal to p^n ($n \geq 1$). By remark above the group G is isomorphic to a fundamental group of a group graph and its vertex subgroups can be embedded into the cyclic group of order p^n . D. Dayer [8] showed that in this case there is a homomorphism ψ of G onto fundamental group $H = \pi(\mathcal{H}, \Gamma)$ of a group graph (\mathcal{H}, Γ) such that

- 1) the graph Γ has only two vertexes u and v ;
- 2) the vertex groups H_u and H_v are cyclic of order p^n and they are generated by elements $x = a\varphi$ and $y = b\varphi$ respectively;
- 3) the order of every edge group H_e is less than p^n .

Thus the generators of H are x , y and t_e , where edge $e \in E(\Gamma)$ is not equal to some fixed edge, the relations of H are:

- a) $x^{p^n} = 1$, $y^{p^n} = 1$;
- b) $x^r = y^s$, where elements $x^r \in H_u$ and $y^s \in H_v$ have the same order which is less than p^n ;
- c) $t_e^{-1}h_1t_e = h_2$, where the elements $h_1 \in H_u$ and $h_2 \in H_v$ have the same order which is less than p^n .

By condition 3) every element $x^r, h_1 \in H_u$, $y^s, h_2 \in H_v$ from relations b) and c) belongs to subgroup K_u or K_v respectively, where orders of K_u and K_v are equal to p^k , $k < n$. Therefore relations b) and c) are trivial in H modulo N where N is a normal closure of K_u and K_v . Thus the quotient-group H/N is a free product of finite cyclic groups $H_u/K_u = (x)$ and $H_v/K_v = (y)$, which orders are equal to p^{n-k} , and the set of infinite cyclic groups (t_e) . The composition of ψ , natural homomorphism of H onto H/N and an evident homomorphism of H/N onto direct product of the groups H_u/K_u and H_v/K_v is required homomorphism of G . The theorem 1 is proved.

3. Suppose H and K are groups, A is a subgroup of H , B is a subgroup of K , $\varphi : A \rightarrow B$ is a isomorphism.

Every element x from $G = (H * K; A = B, \varphi)$ can be presented as $x = x_1x_2 \dots x_n$, where each x_1, x_2, \dots, x_n is from factor H or K and if $n > 1$ then x_i and x_{i+1} are from different factors for every $i = 1, \dots, n-1$ (therefore they do not belong to A and B). This presentation is called reduced form of x and the number n (that doesn't depend of choice of such presentation) is called the length of x . An element $x \in G$ is called

cyclically reduced if either its length n equals to 1 or $n > 1$ and elements x_1 and x_n in its reduced form are from different factors H and K . In this case the expression $u_i = x_i x_{i+1} \cdots x_n x_1 \cdots x_{i-1}$ is reduced for each $i = 1, 2, \dots, n$. The element u_i is called a cyclic permutation of x (if $n = 1$ then x is a unique cyclic permutation of x).

When amalgamating subgroups A and B are central then the general conditions for two elements from G to be conjugate ([3]) can be simplified:

Proposition 3.1. *Suppose $G = (H * K; A = B, \varphi)$ is a free product of two groups with amalgamated central subgroups A and B . For every $g \in G$ there is a cyclically reduced $x \in G$ such that g and x are conjugate. Suppose $x \in G$ and $y \in G$ are cyclically reduced. Then x and y are conjugate in G if and only if their length are equal and either they are from one factor H or K and conjugate in it, or their lengths more than 1 and one of these elements equals to the cyclic permutation of another.*

Let's remind also the following notion ([5]). Subgroups $R \leq H$ and $S \leq K$ are called (A, B, φ) -compatible if $(A \cap R)\varphi = B \cap S$. If normal subgroups $R \leq H$ and $S \leq K$ are (A, B, φ) -compatible then the mapping $\varphi_{R,S} : AR/R \rightarrow BS/S$, where $(aR)\varphi_{R,S} = (a\varphi)S$ ($a \in A$), is an isomorphism of subgroup $AR/R \leq H/R$ onto subgroup $BS/S \leq K/S$. Thus there is a free product

$$G_{R,S} = (H/R * K/S; AR/R = BS/S, \varphi_{R,S})$$

of the groups H/R and K/S with amalgamated via isomorphism $\varphi_{R,S}$ subgroups AR/R and BS/S . Natural homomorphisms of the group H onto quotient group H/R and of the group K onto quotient group K/S can be extended to a homomorphism $\rho_{R,S}$ of the group $G = (H * K; A = B, \varphi)$ onto the group $G_{R,S}$.

Proposition 3.2. *Suppose that H and K are conjugacy p -separable groups, $A \leq H$ and $B \leq K$ are central subgroups and subgroups A and B and every finite p -index subgroup of A and B are p -separable in group H and K respectively. Then for every finite p -index normal subgroups $M \leq H$ and $N \leq K$ there are (A, B, φ) -compatible finite p -index normal subgroups $R \leq H$ and $S \leq K$ such that $R \leq M$ and $S \leq N$.*

Proof. Suppose M contains a finite p -index (in A) subgroup $U \leq A$. Then subgroup U is p -separable in the group H and therefore the quotient

group H/U is residually finite p -groups. Also since the quotient group A/U is a finite subgroup of the quotient group H/U , therefore there is a finite p -index normal subgroup R/U of the group H/U such that $R/U \cap A/U = 1$. Then R is a finite p -index normal subgroup of the group H and $R \cap A = U$. We can consider also that $R \leq M$. The similar reasoning is fair and for the group K .

Suppose M and N are finite p -index normal subgroups of the groups H and K respectively, $U = (M \cap A) \cap (N \cap B)\varphi^{-1}$ and $V = (M \cap A)\varphi \cap (N \cap B)$. Then there are finite p -index normal subgroups R and S of the groups M and N respectively such that $R \leq M$, $R \cap A = U$, $KS \leq N$ and $S \cap B = V$. Since $U\varphi = V$ the subgroups R and S are required.

Now we are ready to prove the theorem 2. Suppose H and K are conjugacy p -separable groups, $G = (H * K; A = B, \varphi)$ is a free product of the groups H and K with amalgamated central subgroups A and B .

Using standard methods of the proof for the free product of two groups with amalgamated subgroups to be residually finite p -groups it is easy to receive the following statement:

Proposition 3.3. *Suppose H and K are residually finite p -groups, $A \leq H$ and $B \leq K$ are central subgroups and subgroups A and B and every finite p -index subgroup of A and B are p -separable in group H and K respectively. Then $G = (H * K; A = B, \varphi)$ is a residually finite p -groups.*

It follows from the theorem 1 that if $R \leq H$ and $S \leq K$ are (A, B, φ) -compatible finite p -index normal subgroups then $G_{R,S}$ is a conjugacy p -separable group (it is proved in [9] that $G_{R,S}$ is a residually finite p -groups). Therefore it is enough to prove that whenever $x \in G$ and $y \in G$ are not conjugate in G there are (A, B, φ) -compatible finite p -index normal subgroups $R \leq H$ and $S \leq K$ such that $x\rho_{R,S}$ and $y\rho_{R,S}$ are not conjugate in $G_{R,S}$.

Suppose $f \in G$ and $g \in G$ are not conjugate in G . Since proposition 3.1 we can assume without generality loss that f and g are cyclically reduced. Let's consider some cases.

Case 1. The lengths of f and g are not equal.

Since the subgroups A and B are p -separable in the groups H and K respectively there are finite p -index normal subgroups $M \leq H$ and $N \leq K$ such that all factors in the reduced forms of f and g (it is fair

only for the elements of length 1) are not from AM and BN respectively. Since proposition 3.2 there are (A, B, φ) -compatible finite p -index normal subgroups $R \leq H$ and $S \leq K$ such that $R \leq M$ and $S \leq N$. Then $f\rho_{R,S}$ and $g\rho_{R,S}$ are cyclically reduced in the group $G_{R,S}$, the lengths of $f\rho_{R,S}$ and $g\rho_{R,S}$ are equal to the lengths of f and g respectively and different. Therefore since proposition 3.1 $f\rho_{R,S}$ and $g\rho_{R,S}$ are not conjugate in $G_{R,S}$.

Case 2. The lengths of f and g are equal to 1, f and g are from different factors H and K .

Suppose $f \in H \setminus A$ and $g \in K \setminus B$. Since A and B are p -separable subgroups in H and K respectively there are finite p -index normal subgroups $M \leq H$ and $N \leq K$ such that $f \notin AM$ and $g \notin BN$. At that time since proposition 3.2 there are (A, B, φ) -compatible finite p -index normal subgroups $R \leq H$ and $S \leq K$ such that $R \leq M$ and $S \leq N$. Then $f\rho_{R,S}$ and $g\rho_{R,S}$ are from different factors H/R and K/S of $G_{R,S}$ and since proposition 3.1 the elements $f\rho_{R,S}$ and $g\rho_{R,S}$ are not conjugate in $G_{R,S}$.

Case 3. The lengths of f and g are equal to 1, f and g are both from one factor H or K .

Suppose $f \in H$ and $g \in H$. Since f and g are not conjugate in H and H is a conjugacy p -separable group there is a finite p -index normal subgroup $M \leq H$ such that fM and gM are not conjugate in H/M . At that time since proposition 3.2 there are (A, B, φ) -compatible finite p -index normal subgroups $R \leq H$ and $S \leq K$ such that $R \leq M$. Then $f\rho_{R,S}$ and $g\rho_{R,S}$ are both from the factor H/R of the group $G_{R,S}$ and the elements $f\rho_{R,S}$ and $g\rho_{R,S}$ are not conjugate in H/R . Therefore since proposition 3.1 $f\rho_{R,S}$ and $g\rho_{R,S}$ are not conjugate in $G_{R,S}$.

Case 4. The lengths of f and g are equal and more than 1.

Suppose g_1, g_2, \dots, g_r (r is a length of g) are all cyclic permutations of g . Since the subgroups A and B are p -separable in the groups H and K respectively there are finite p -index normal subgroups $R_0 \leq H$ and $S_0 \leq K$ such that all factors in the reduced forms of f and g are not from AR_0 and BS_0 respectively. Since f is not equal to g_1, g_2, \dots, g_r and G is a residually finite p -groups (proposition 3.3) there is a finite p -index normal subgroup $N \leq G$ such that fN is not equal to g_1N, g_2N, \dots, g_rN . Suppose $R = R_0 \cap N$ and $S = S_0 \cap N$. Then $f\rho_{R,S}$ and $g\rho_{R,S}$ are cyclically reduced in $G_{R,S}$, their lengths are equal to r and $f\rho_{R,S}$ is not equal to $g_1\rho_{R,S}, g_2\rho_{R,S}, \dots, g_r\rho_{R,S}$. Since every cyclic permutation

of $g\rho_{R,S}$ is equal to one of the elements $g_1\rho_{R,S}, g_2\rho_{R,S}, \dots, g_r\rho_{R,S}$ the elements $f\rho_{R,S}$ and $g\rho_{R,S}$ are not conjugate in $G_{R,S}$ (proposition 3.1).

The theorem 2 is proved.

Since every p' -isolated subgroup of a finitely generated nilpotent group is p -separable the theorem 3 immediately follows from the theorem 2.

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